Conservation Laws in "Doubly Special Relativity"

Simon Judes*
Columbia University, New York, NY 10027, U.S.A.

Matt Visser†
School of Mathematical and Computing Sciences,
Victoria University of Wellington, New Zealand

Motivated by various theoretical arguments that the Planck energy ($E_{\mathrm{Planck}} \sim 10^{19}$ GeV) should herald departures from Lorentz invariance, and the possibility of testing these expectations in the not too distant future, two so-called "Doubly Special Relativity" theories have been suggested — the first by Amelino-Camelia (DSR1) and the second by Smolin and Magueijo (DSR2). These theories contain two fundamental scales — the speed of light and an energy usually taken to be E_{Planck}. The symmetry group is still the Lorentz group, but in both cases acting nonlinearly on the energy-momentum sector. Since energy and momentum are no longer additive quantities, finding their values for composite systems (and hence finding appropriate conservation laws) is a nontrivial matter. Ultimately it is these possible deviations from simple linearly realized relativistic kinematics that provide the most promising observational signal for empirically testing these models. Various investigations have narrowed the conservation laws down to two possibilities per DSR theory. We derive unique exact results for the energy-momentum of composite systems in both DSR1 and DSR2, and indicate the general strategy for arbitrary nonlinear realizations of the Lorentz group.

Background: Observations of very high energy cosmic rays, above the expected "GZK cutoff" due to interaction with microwave background radiation [1, 2], have precipitated a surge of interest in possible violations of Lorentz invariance. Encouragingly it appears that this phenomenon may furnish experimental tests of some suggested theories of quantum gravity [3, 4, 5, 6, 7, 8]. For a review, see [9]. One type of Lorentz violating theory is known as "Doubly Special Relativity" after Amelino-Camelia [10], who has suggested a specific example of a DSR theory [11]. Smolin and Magueijo have suggested another [12] in a paper in which they argued that any DSR transformation group *must* be a nonlinear realization of the Lorentz group — because that is the only suitable 6 parameter extension of SO(3) — the group of spatial rotations. Unlike ordinary special relativity, in DSR the transformation properties of energy and momentum need not be the same as those of the space-time coordinates. Many investigations have been limited to the energy-momentum sector [10, 11]. One approach that deals with space-time as well (it it presently unclear if there are others) is in terms of the κ–Poincaré algebra — a deformation of the Poincaré algebra [13, 14]. The algebras obeyed by the DSR1 and DSR2 Lorentz generators are known to be just such nonlinear deformations [12, 15, 16] of the κ–Lorentz subalgebra — DSR1 corresponding to the so-called "bi-crossproduct basis". Because there is still some controversy and uncertainty regarding the issue of whether or not all DSR theories are *necessarily* κ–Poincaré theories, we will stay in momen-

tum space and deal only with general features of arbitrary nonlinear representations of the Lorentz group [17].

To find conservation laws, two distinct approaches have been used. One method [15, 16] is to investigate the nature of the nonlinear realization of the symmetry group instantiated by the DSR transformations and use its properties as constraints on the conservation laws for composite systems. The alternative [10, 11] is to work directly with the transformation equations and to apply physically intuitive restrictions to deduce the laws. Through a combination of these two techniques, the number of possible conservation laws for DSR1 and DSR2 has been reduced to two. We continue along the lines of the second method, and find that it is possible to uniquely identify the conservation laws for *any* DSR theory by applying seemingly reasonable physical principles. We give exact results for the total energy and momentum of a composite system in both DSR1 and DSR2. Because these formulae implicitly control particle production thresholds they are critically important in assessing phenomenological attempts to place observational constraints on the DSR theories [9, 18, 19, 20, 21].

General rules: Since a DSR symmetry group is simply a nonlinear realization of the Lorentz group [12, 15, 16], we can find functions of the physical energy-momentum $P_4 = (E, p)$ which transform like a Lorentz 4–vector. These we will call the pseudo-energy-momentum $\mathcal{P}_4 = (\epsilon, \pi)$, but it should not be thought that these necessarily have immediate physical significance. We have:

$$P_4 = F(\mathcal{P}_4); \qquad \mathcal{P}_4 = F^{-1}(P_4). \qquad (1)$$

The function F and its inverse F^{-1} are in general complicated nonlinear functions from $\Re^4$ to $\Re^4$, but both of course reduce to the identity in the limit where energies and momenta are small compared to the DSR scale. The Lorentz transformations act on the auxiliary variables in the normal linear manner: $(\epsilon'; \pi') = \mathcal{L}\,(\epsilon; \pi)$. where $\mathcal{L}$ is the usual Lorentz transformation, boosting from the unprimed coordinates to the primed coordinates. The boost operator for the physical energy and momentum (E, p) we call L, and is given by the composition:

$$P_4' = L(P_4) = [\, F \circ \mathcal{L} \circ F^{-1}\,](P_4). \tag{2}$$

Now $\mathcal{L}$ and F uniquely determine the nonlinear Lorentz transformation L; however $\mathcal{L}$ and L [more precisely, $L(\mathcal{L})$] do not uniquely determine the function F — there is an overall multiplicative ambiguity which must be dealt with using the dispersion relation:

$$[\epsilon(E, p)]^2 - [\pi(E, p)]^2 = \mu_0^2. \tag{3}$$

Here μ_0 is simply the Lorentz invariant constructed from ϵ and π (the Casimir invariant); not to be confused with the rest energy. In terms of the rest energy m_0, obtained by going to a Lorentz frame in which the particle is at rest, $\mu_0 = \epsilon(m_0, 0)$. The combination of $L(\mathcal{L})$ and $\mu_0(m_0)$ is now sufficient to pin down F completely.

In the linear representation, kinematic quantities such as total energy can be defined in the usual fashion

$$\mathcal{P}_4^{tot} = \sum_i \mathcal{P}_4^i. \tag{4}$$

Calculating the total physical 4-momentum is then straightforward:

$$P_4^{tot} = F\left(\sum_i F^{-1}(P_4^i)\right). \tag{5}$$

This is the quantity that will be conserved in collisions. Calculating it is simply a matter of finding F and its inverse.

Variant conservation laws: The choice in equation (4), and so implicitly in equation (5), can be uniquely characterized by saying that the general composition of 4-momenta is based on iterating an associative symmetric binary function.

If the general composition law were not based on iterating a binary function, then one would need to postulate an infinite tower of distinct composition laws for $2, 3, 4, \ldots, n, \ldots$ particles. Such a situation would create serious difficulties in the interpretation of quantum field theories: For instance, energy-momentum conservation at each vertex of a Feynman diagram would now depend in an essentially arbitrary way on a particular time-slice through the diagram and the energy-momenta of all other particles in the diagram as they cross that time-slice. Perhaps worse, every time a dressed particle were to either emit or absorb a virtual particle one would have to completely recalculate the energy-momentum for the entire virtual cloud.

If the binary function were not symmetric, one could (simply by changing the order in which one chooses to list the particles) construct symmetric and anti-symmetric combinations, leading to two separate conservation laws that would over-constrain the collision (unless, of course, the anti-symmetric law happens to be trivial — but that implies a symmetric binary function).

Finally, if the binary function were not associative, then the energy-momentum of a composite system would depend not only on the constituents of the system, but also on the manner in which the system is aggregated out of subsystems — an option that is at best extremely unnatural.

The initial investigations into energy and momentum of composite systems in DSR [11] proceeded only on the requirement that the law of energy-momentum conservation had to be covariant with respect to the DSR transformation. The insufficiency of this requirement is manifest when we consider that the following definition:

$$\mathcal{P}_4^{tot} = \sum_i \nu_i\, \mathcal{P}_4^i. \tag{6}$$

produces a covariant conservation law for *arbitrary* ν_i. Symmetry, which is required to prevent over-determining the energy-momentum in a collision, implies that:

$$\mathcal{P}_4^{tot} = \nu \sum_i \mathcal{P}_4^i. \tag{7}$$

If this is to arise from iterating a two-particle composition law we need $\mathcal{P}_4^{\{12\}} = \nu(\mathcal{P}_4^1 + \mathcal{P}_4^2)$. But now for a three-particle system, associativity implies

$$\nu[\nu(\mathcal{P}_4^1 + \mathcal{P}_4^2) + \mathcal{P}_4^3] = \nu[\mathcal{P}_4^1 + \nu(\mathcal{P}_4^2 + \mathcal{P}_4^3)] \tag{8}$$

Therefore $\nu = \nu^2$, implying either $\nu = 1$ or $\nu = 0$. This argument gives the same result as that used by Lukierski and Nowicki [16] to reduce the number of possible laws to two. In fact, their "symmetric" and "non-symmetric" laws are just the $\nu = 1$ and $\nu = 0$ cases respectively. The $\nu = 1$ solution is clearly unproblematic. However, what is not evident from the group theoretic analysis of [16], and is evident from the current approach, is the rather odd nature of the case where $\nu = 0$. Taken straightforwardly, it must be false, implying that for any number of particles

$$\mathcal{P}_4^{tot} = \vec{0}; \qquad P_4^{tot} = F(\vec{0}). \tag{9}$$

Thus $\nu = 0$ is clearly unphysical and we are forced to adopt the intuitive choice of $\nu = 1$.

We feel that more drastic possibilities [17], based on abandoning notions of an iterated associative symmetric binary composition law are strongly disfavoured, and will not pursue such options in this Letter.

DSR2: This model [12] is completely characterized by the equation

$$\mathcal{P}_4 \equiv (\epsilon; \pi) = F^{-1}(P_4) = \frac{(E; p)}{1 - \lambda E}. \tag{10}$$

(In model building one typically takes $\lambda = 1/E_{\text{Planck}}$; but we will leave λ as an arbitrary parameter with dimensions $[E]^{-1}$.) The inverse mapping is easily established to be

$$P_4 \equiv (E; p) = F(\mathcal{P}_4) = \frac{(\epsilon; \pi)}{1 + \lambda \epsilon}. \tag{11}$$

The total physical 4-momentum is easily calculated. First observe that for the pseudo-momenta

$$\epsilon_{tot} = \sum_i \frac{E_i}{1 - \lambda E_i}; \qquad \pi_{tot} = \sum_i \frac{p_i}{1 - \lambda E_i}. \tag{12}$$

Then

$$E_{tot} = \frac{\sum_i E_i/(1 - \lambda E_i)}{1 + \lambda \sum_i E_i/(1 - \lambda E_i)}. \tag{13}$$

and

$$p_{tot} = \frac{\sum_i p_i/(1 - \lambda E_i)}{1 + \lambda \sum_i E_i/(1 - \lambda E_i)}. \tag{14}$$

Within the framework of DSR2 this result is *exact* for all λ. To first-order in λ:

$$E_{tot} = \sum_i E_i - \lambda \sum_{i \neq j} E_i E_j + O(\lambda^2), \tag{15}$$

$$p_{tot} = \sum_i p_i - \lambda \sum_{i \neq j} p_i E_j + O(\lambda^2). \tag{16}$$

For the case of two particles, the above formulae reduce to the so-called 'mixing laws' — one of the possibilities mentioned by Amelino-Camelia *et al.* [10].

We also mention in passing that the exact dispersion relation for DSR2 is

$$\frac{E^2 - p^2}{(1 - \lambda E)^2} = \mu_0^2 = \frac{m_0^2}{(1 - \lambda m_0)^2}. \tag{17}$$

This can be rearranged as

$$p^2 = E^2 - m_0^2 \left(\frac{1 - \lambda E}{1 - \lambda m_0} \right)^2. \tag{18}$$

Solving the quadratic for E, and choosing the physical root

$$E = \frac{\sqrt{(1 - 2\lambda m_0)[m_0^2 + (1 - \lambda m_0)^2 p^2] + \lambda^2 m_0^4} - \lambda m_0^2}{1 - 2\lambda m_0}. \tag{19}$$

DSR1: For DSR1 the basic principles are the same but the algebra is somewhat messier. It is convenient to consider a particle at rest, with rest energy m_0, and then boost using a rapidity parameter ξ. The defining relationships for DSR1 can then be put in the form [11]

$$e^{\lambda E} = e^{\lambda m_0} \left(1 + \sinh(\lambda m_0) \, e^{-\lambda m_0} \left[\cosh \xi - 1 \right] \right), \tag{20}$$

and

$$p = \frac{1}{\lambda} \frac{\sinh(\lambda m_0) \, e^{-\lambda m_0} \, \sinh \xi}{1 + \sinh(\lambda m_0) \, e^{-\lambda m_0} \left[\cosh \xi - 1 \right]}. \tag{21}$$

(These expressions are equivalent to knowing the nonlinear Lorentz transformations L as a function of rapidity ξ.) This can easily be inverted to give expressions for the rapidity

$$\cosh \xi = \frac{e^{\lambda E} - \cosh(\lambda m_0)}{\sinh(\lambda m_0)}; \quad \sinh \xi = \frac{\lambda \, p \, e^{\lambda E}}{\sinh(\lambda m_0)}. \tag{22}$$

Making use of the identity $\cosh^2 \xi - \sinh^2 \xi = 1$ gives the DSR1 dispersion relation in the particularly nice form

$$\cosh(\lambda E) = \cosh(\lambda m_0) + \frac{1}{2} \lambda^2 p^2 e^{\lambda E}. \tag{23}$$

Comparison with the standard form of the dispersion relation now fixes the rest energy in terms of the Casimir invariant

$$\cosh(\lambda m_0) = 1 + \frac{1}{2} \lambda^2 \mu_0^2; \qquad \mu_0 = \frac{2 \sinh(\lambda m_0/2)}{\lambda}. \tag{24}$$

This now fixes the linear representation completely. In terms of the physical energy-momenta

$$\epsilon = \mu_0 \cosh \xi = \frac{e^{\lambda E} - \cosh(\lambda m_0)}{\lambda \cosh(\lambda m_0/2)}, \tag{25}$$

and

$$\pi = \mu_0 \sinh \xi = \frac{p \, e^{\lambda E}}{\cosh(\lambda m_0/2)}. \tag{26}$$

Conversely, the inverse mappings yielding physical energy-momenta in terms of auxiliary energy-momenta are

$$\begin{aligned}
E &= \frac{1}{\lambda} \ln \left[\lambda \epsilon \cosh(\lambda m_0/2) + \cosh(\lambda m_0) \right], \\
&= \frac{1}{\lambda} \ln \left[1 + \lambda \epsilon \sqrt{1 + \frac{\lambda^2 \mu_0^2}{4}} + \frac{\lambda^2 \mu_0^2}{2} \right], \\
&= \frac{1}{\lambda} \ln \left[1 + \lambda \epsilon \sqrt{1 + \frac{\lambda^2 (\epsilon^2 - \pi^2)}{4}} + \frac{\lambda^2 (\epsilon^2 - \pi^2)}{2} \right],
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
p &= \pi \cosh(\lambda m_0/2) e^{-\lambda E}, \\
&= \frac{\pi \sqrt{1 + \frac{\lambda^2 \mu_0^2}{4}}}{1 + \lambda \epsilon \sqrt{1 + \frac{\lambda^2 \mu_0^2}{4}} + \frac{\lambda^2 \mu_0^2}{2}}, \\
&= \frac{\pi \sqrt{1 + \frac{\lambda^2 (\epsilon^2 - \pi^2)}{4}}}{1 + \lambda \epsilon \sqrt{1 + \frac{\lambda^2 (\epsilon^2 - \pi^2)}{4}} + \frac{\lambda^2 (\epsilon^2 - \pi^2)}{2}}.
\end{aligned} \tag{28}$$

To calculate total energy and momentum of a collection of particles we now first calculate auxiliary quantities

$$\epsilon_{tot} = \sum_i \frac{e^{\lambda E_i} - \cosh(\lambda m_{0,i})}{\lambda \cosh(\lambda m_{0,i}/2)}, \qquad (29)$$

$$\pi_{tot} = \sum_i \frac{p_i \, e^{\lambda E_i}}{\cosh(\lambda m_{0,i}/2)}, \qquad (30)$$

and then use these to calculate the physical quantities

$$E_{tot} = \frac{1}{\lambda} \ln \left[1 + \lambda \epsilon_{tot} \sqrt{1 + \frac{\lambda^2(\epsilon_{tot}^2 - \pi_{tot}^2)}{4}} + \frac{\lambda^2(\epsilon_{tot}^2 - \pi_{tot}^2)}{2} \right], \qquad (31)$$

$$p_{tot} = \frac{\pi_{tot}\sqrt{4 + \lambda^2(\epsilon_{tot}^2 - \pi_{tot}^2)}}{2 + \lambda \epsilon_{tot}\sqrt{4 + \lambda^2(\epsilon_{tot}^2 - \pi_{tot}^2)} + \lambda^2(\epsilon_{tot}^2 - \pi_{tot}^2)}. \qquad (32)$$

These formulae provide explicit (albeit complicated) expressions for the total physical energy and momenta in the DSR1 model in terms of the individual physical energy, momenta, and rest energies; note that the formulae are *exact* for arbitrary λ.

To first order:

$$E_{tot} = \sum_i E_i - \frac{1}{2}\lambda \sum_{i \neq j} p_i p_j + O(\lambda^2), \qquad (33)$$

$$p_{tot} = \sum_i p_i - \lambda \sum_{i \neq j} p_i E_j + O(\lambda^2). \qquad (34)$$

For two particles, these too reduce to equations already in the literature [11, 16].

Discussion: The key result of this note is the identification of appropriate laws of conservation of energy and momentum in generic DSR theories, embodied in the general formula (5), together with the specific applications to DSR2 in equations (13)—(14), and to DSR1 in equations (29)—(32). Ultimately the general formula (5) is more important: There are many ways of distorting the Lorentz group, and this formula applies to all of them — this makes it clear that the distortion of dispersion relations, the existence of unexpected thresholds, and the somewhat unexpected subtleties hiding in the conservation laws are generic to all nonlinear realizations of the Lorentz group, no matter how they are obtained. It is these possible deviations from simple linearly realized relativistic kinematics that provide the most promising observational signal for empirically testing these models [9, 18, 19].

* Electronic address: sj2057@columbia.edu
† Electronic address: matt.visser@vuw.ac.nz
[1] K. Greisen, Phys. Rev. Lett. **16** (1966) 748.
[2] G. T. Zatsepin and V. A. Kuzmin, JETP Lett. **4** (1966) 78 [Pisma Zh. Eksp. Teor. Fiz. **4** (1966) 114].
[3] G. Amelino-Camelia, J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos and S. Sarkar, Nature **393** (1998) 763 [arXiv:astro-ph/9712103].
[4] S. R. Coleman and S. L. Glashow, Phys. Rev. D **59** (1999) 116008 [arXiv:hep-ph/9812418].
[5] O. Bertolami and C. S. Carvalho, Phys. Rev. D **61** (2000) 103002 [arXiv:gr-qc/9912117].
[6] T. Kifune, Astrophys. J. **518** (1999) L21 [arXiv:astro-ph/9904164].
[7] W. Kluzniak, arXiv:astro-ph/9905308.
W. Kluzniak, Astropart. Phys. **11** (1999) 117.
[8] G. Amelino-Camelia and T. Piran, Phys. Lett. B **497** (2001) 265 [arXiv:hep-ph/0006210].
G. Amelino-Camelia and T. Piran, Phys. Rev. D **64** (2001) 036005 [arXiv:astro-ph/0008107].
[9] S. Sarkar, Mod. Phys. Lett. A **17** (2002) 1025 [arXiv:gr-qc/0204092].
[10] G. Amelino-Camelia, D. Benedetti and F. D'Andrea, "Comparison of relativity theories with observer-independent scales of both velocity and length/mass," arXiv:hep-th/0201245.
[11] G. Amelino-Camelia, Int. J. Mod. Phys. D **11** (2002) 35 [arXiv:gr-qc/0012051].
G. Amelino-Camelia, Phys. Lett. B **510** (2001) 255 [arXiv:hep-th/0012238].
[12] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88** (2002) 190403 [arXiv:hep-th/0112090].
[13] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B **264** (1991) 331
[14] S. Majid and H. Ruegg, Phys. Lett. B **334** (1994) 348 [arXiv:hep-th/9405107].
[15] J. Kowalski-Glikman and S. Nowak, Phys. Lett. B **539** (2002) 126 [arXiv:hep-th/0203040].
[16] J. Lukierski and A. Nowicki, "Doubly Special Relativity versus κ-deformation of relativistic kinematics," arXiv:hep-th/0203065.
[17] G. Amelino-Camelia, Nature **418** (2002) 34 [arXiv:gr-qc/0207049].
[18] T. Jacobson, S. Liberati and D. Mattingly, Phys. Rev. D **66** (2002) 081302 [arXiv:hep-ph/0112207].
"Threshold effects and Planck scale Lorentz violation: Combined constraints from high energy astrophysics," arXiv:hep-ph/0209264.
"Threshold configurations in the presence of Lorentz violating dispersion relations," arXiv:hep-ph/0211466.
[19] S. Liberati, T. A. Jacobson and D. Mattingly, "High energy constraints on Lorentz symmetry violations," arXiv:hep-ph/0110094.
[20] G. Amelino-Camelia, Mod. Phys. Lett. A **17** (2002) 899 [arXiv:gr-qc/0204051].
[21] T. J. Konopka and S. A. Major, New J. Phys. **4** (2002) 57 [arXiv:hep-ph/0201184].